A Parameterized Post-Friedmann Framework for Modified Gravity

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We develop a parameterized post-Friedmann (PPF) framework which describes three regimes of modified gravity models that accelerate the expansion without dark energy. On large scales, the evolution of scalar metric and density perturbations must be compatible with the expansion history defined by distance measures. On intermediate scales in the linear regime, they form a scalar-tensor theory with a modified Poisson equation. On small scales in dark matter halos such as our own galaxy, modifications must be suppressed in order to satisfy stringent local tests of general relativity. We describe these regimes with three free functions and two parameters: the relationship between the two metric fluctuations, the large and intermediate scale relationships to density fluctuations and the two scales of the transitions between the regimes. We also clarify the formal equivalence of modified gravity and generalized dark energy. The PPF description of linear fluctuation in $f(R)$ modified action and the Dvali-Gabadadze-Porrati braneworld models show excellent agreement with explicit calculations. Lacking cosmological simulations of these models, our non-linear halo-model description remains an ansatz but one that enables well-motivated consistency tests of general relativity. The required suppression of modifications within dark matter halos suggests that the linear and weakly non-linear regimes are better suited for making complementary test of general relativity than the deeply non-linear regime.

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I. INTRODUCTION

Theoretically compelling alternatives to a cosmological constant as the source of the observed cosmic acceleration are currently lacking. In the absence of such alternatives, it is useful to have a phenomenological parameterized approach for testing the predictions of a cosmological constant and phrasing constraints in a model-independent language. This approach parallels that of local tests of general relativity. The parameterized post-Newtonian description of gravity forms a complete description of leading order deviations from general relativity locally under a well-defined set of assumptions [1].

A parameterization of cosmic acceleration from the standpoint of dark energy is now well-established. The expansion history that controls distance observables is completely determined by the current dark energy density and its equation of state as a function of redshift. Structure formation tests involve additional parameters that control inhomogeneities in the dark energy. Covariant conservation of energy-momentum requires that the dark energy respond to metric or gravitational potential fluctuations at least on scales above the horizon. In a wide class of models where the dark energy remains smooth relative to the matter on small scales, the phenomenological parameter of interest is where this transition occurs [2, 3, 4].

A similar structure is imposed on modified gravity models that accelerate the expansion without dark energy. Requirements that gravity remain a metric theory where energy-momentum is covariantly conserved also place strong constraints their scalar degrees of freedom. On scales above the horizon, structure evolution must be compatible with the background expansion [5]. Intermediate scales are characterized by a scalar-tensor theory with a modified Poisson equation [6]. If these modifications are to pass stringent local tests of gravity then additional scalar degrees of freedom must be suppressed locally [7]. Two explicit models that exhibit all three regimes of modified gravity are the so-called $f(R)$ modified Einstein-Hilbert action models [8, 9, 10] and the Dvali-Gabadadze-Porrati (DGP) braneworld model [11].

Although several parameterized gravity approaches exist in the literature, none describe all three regimes of modified gravity (cf. [12, 13]) and most do not explicitly enforce a metric structure to gravity or energy momentum conservation (e.g. [14, 15, 16, 17, 18]).

In this paper, we develop a parameterized post-Friedmann (PPF) framework that describes all three regimes of modified gravity models that accelerate the expansion without dark energy. We begin in §II by describing the three regimes individually and the requirements they impose on the structure of such modifications. In §III, we describe a linear theory parameterization of the first two regimes and test it against explicit calculations of the $f(R)$ and DGP models. In §IV, we develop a non-linear ansatz for the third regime based on the halo model of non-linear clustering. In the Appendix, we clarify the formal relationship between modified gravity and dark energy beyond the smooth class of models.

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flecting the opposite sign of g in the linear regime of the two models.

V. DISCUSSION

We have introduced a parameterized framework for considering scalar modifications to gravity that accelerate the expansion without dark energy. This framework features compatibility in the evolution of structure with a background expansion history on large scales, a modification of the Poisson equation on intermediate scales, and a return to general relativity within collapsed dark matter halos. This return to general relativity is required of models to pass stringent local tests of gravity. We have also clarified the formal relationship between modified gravity and dark energy in the Appendix. A metric based modified gravity model can always be cast in terms of a dark energy component with a stress energy tensor defined to match its influence on the metric. However such a component would possess dynamics which are coupled to the matter.

Our parameterized post-Friedmann framework features several free functions even in the linear regime. The most important function is the relationship $g = (\Phi + \Psi)/(\Phi - \Psi)$ between the time-time and space-space pieces of the metric in Newtonian gauge. Supplementing these are two functions that link the metric to matter density perturbations: one on super-horizon scales and one on intermediate scales. Finally there is a parameter that controls the interpolation between these two regimes.

We have shown that with an appropriate choice of parameters this framework describes linear perturbations in the $f(R)$ modified action and DGP braneworld gravity models. It may be used in place of the more complicated 4th order and higher dimensional dynamics exhibited in these models respectively when studying phenomena such as the integrated Sachs-Wolfe effect in the CMB, large-scale gravitational lensing and galaxy clustering. We intend to explore these applications in a future work.

On non-linear scales our framework features an ansatz based on the requirement that scalar modifications should be suppressed locally in order to pass the stringent tests of general relativity in the solar system. Indeed the scalar degrees of freedom in both the $f(R)$ and the DGP models possess non-linearities that drive the dynamics back to general relativity in high curvature or high density regimes. Our ansatz is based on the halo model of non-linear clustering. It allows for a density dependent interpolation for the abundance and structure of dark matter halos between the expectations of general relativity and the modified Poisson equation on intermediate scales.

Due to the current lack of cosmological simulations in these modified gravity models, the accuracy of our simple ansatz remains untested. With cosmological simulations, our framework can be extended and refined by introducing more parameters that describe the potentially mass-dependent modification of dark matter haloes. In fact, our simple halo model parameterization is not even sufficient to accurately model non-linear effects in general relativity. Nonetheless phrased as a simple template form for relative deviations in the power spectrum between modified gravity and general relativity with smooth dark energy, our current ansatz can be used in conjunction with more accurate results from dark energy cosmological simulations. For example, it can be used to search for possible deviations of this type as a consistency check on dark energy inferences from expansion history tests with upcoming cosmic shear surveys.

While many such consistency tests have been proposed in the literature, it is important to incorporate a density dependence to the modifications as we have done here. The principle that non-linear scales should exhibit a return to general relativity itself suggests that mildly non-linear scales provide the most fruitful window for cosmological tests of gravity. Furthermore uncertainties in the baryonic influence on the internal structure of dark matter halos in the deeply non-linear regime even under general relativity (e.g. [37, 38]) make consistency tests in this regime potentially ambiguous. Our parameterized framework should enable studies of such issues in the future.

APPENDIX A: DARK ENERGY CORRESPONDENCE

Suppose we view the modifications to gravity in terms of an additional "dark energy" stress tensor. We are free to *define* the dark energy stress tensor to be

$$T_e^{\mu\nu} \equiv \frac{1}{8\pi G} G^{\mu\nu} - T_m^{\mu\nu} . \tag{A1}$$

Given this association, all of the familiar structure of cosmological perturbation theory in general relativity applies. In particular, covariant conservation of the matter stress energy tensor $T_m^{\mu\nu}$ and the Bianchi identities imply conservation of the effective dark energy

$$\nabla_\mu T_e^{\mu\nu} = 0 . \tag{A2}$$

The remaining degrees of freedom in the effective dark energy stress tensor can then be parameterized in the same manner as a general dark energy component [3]. Two models that imply the same $T_e^{\mu\nu}$ at all points in spacetime are formally indistinguishable gravitationally [21, 39].

Note however that this equivalence is only formal and two physically distinct models, e.g. $f(R)$ modified gravity and scalar field dark energy, will not in general imply the same effective stress energy tensor. The Einstein and conservation equations do not form a closed system and the distinction between modified gravity and dark energy lies in the closure relation. For dark energy that

8 is not coupled to matter, the closure relationship takes the form of equations of state that define its internal dynamics. These micro-physical relations do not depend explicitly on the matter. For example for scalar field dark energy, the sound speed or the relationship between the pressure and energy density fluctuations is defined in the constant field gauge without reference to the matter [3], and is associated with the form of the kinetic term in the Lagrangian [40].

8 For modified gravity of the type described in this paper, we shall see that the closure relations must depend explicitly on the matter. The effective dark energy of a modified gravity model must be coupled to the matter. In other words, while the modification gravity can be modeled as fifth forces mediated by the effective dark energy, it cannot be viewed as a missing energy component that obeys separate equations of motion.

It is nonetheless useful to phrase the PPF parameterization in terms of an effective dark energy component. It enables the use of the extensive tools developed for cosmological perturbation theory and facilitates the development of PPF formalisms in different gauges.

1. Covariant Field and Conservation Equations

Following [20, 41], we parameterize linear scalar metric fluctuations of a comoving wavenumber k as

$$
\begin{aligned}
g^{00} &= -a^{-2}(1-2AY)\,, \\
g^{0i} &= -a^{-2}BY^i\,, \\
g^{ij} &= a^{-2}(\gamma^{ij} - 2H_L Y\gamma^{ij} - 2H_T Y^{ij})\,,
\end{aligned} \tag{A3}
$$

where the "0" component denotes conformal time $\eta = \int dt/a$ and γ_{ij} is the background spatial metric which we assume to be flat across scales comparable to the wavelength. Under this assumption, the spatial harmonics are simply plane waves

$$
\begin{aligned}
Y &= e^{i\mathbf{k}\cdot\mathbf{x}}\,, \\
Y_i &= (-k)\nabla_i Y\,, \\
Y_{ij} &= (k^{-2}\nabla_i\nabla_j + \gamma_{ij}/3)Y\,.
\end{aligned} \tag{A4}
$$

Likewise the components of the stress tensors can be parameterized as

$$
\begin{aligned}
T^0{}_0 &= -\rho - \delta\rho\,, \\
T_0{}^i &= -(\rho+p)vY^i\,, \\
T^i{}_j &= (p+\delta p Y)\delta^i{}_j + p\Pi Y^i{}_j\,,
\end{aligned} \tag{A5}
$$

where we will use the subscripts m to denote the matter and e to denote the effective dark energy. When no subscript is specified we mean the components of the total or matter plus effective dark energy stress tensor. For simplicity we assume that the radiation is negligible during the epochs of interest.

By definition, Eqn. (A1) enforces the usual 4 Einstein field equations [21]

$$
\begin{aligned}
&H_L + \frac{1}{3}H_T + \frac{B}{k_H} - \frac{H_T'}{k_H^2} \\
&\qquad = \frac{4\pi G}{H^2 k_H^2}\left[\delta\rho + 3(\rho+p)\frac{v-B}{k_H}\right]\,, \\
&A + H_L + \frac{H_T}{3} + \frac{B'+2B}{k_H} - \left[\frac{H_T''}{k_H^2} + \left(3+\frac{H'}{H}\right)\frac{H_T'}{k_H^2}\right] \\
&\qquad = -\frac{8\pi G}{H^2 k_H^2}p\Pi\,, \\
&A - H_L' - \frac{H_T'}{3} = \frac{4\pi G}{H^2}(\rho+p)\frac{v-B}{k_H}\,, \\
&A' + \left(2 + 2\frac{H'}{H} - \frac{k_H^2}{3}\right)A - \frac{k_H}{3}(B'+B) \\
&\qquad - H_L'' - \left(2+\frac{H'}{H}\right)H_L' = \frac{4\pi G}{H^2}(\delta p + \frac{1}{3}\delta\rho)\,,
\end{aligned} \tag{A6}
$$

where recall $' = d/d\ln a$ and $k_H = (k/aH)$. The conservation laws for the matter and effective dark energy become

$$
\begin{aligned}
&\delta\rho' + 3(\delta\rho + \delta p) = -(\rho+p)(k_H v + 3H_L')\,, \\
&\frac{[a^4(\rho+p)(v-B)]'}{a^4 k_H} = \delta p - \frac{2}{3}p\Pi + (\rho+p)A\,.
\end{aligned} \tag{A7}
$$

There are 4 metric variables and 4 matter variables per component that obey 4 Einstein equations and 2 conservation equations per component. However 2 out of 4 of the Einstein equations are redundant since the Bianchi identities are automatically satisfied given a metric. Furthermore, 2 degrees of freedom simply represent gauge or coordinate freedom. This leaves 2 degrees of freedom per component to be specified. Usually, this involves defining equations of state that specify the spatial stresses in terms of the energy density and velocities. As we shall see, it is this prescription that must be altered to describe modified gravity.

2. Gauge

The scalar gauge degrees of freedom are fixed by gauge conditions. Under a gauge transformation defined by the change in conformal time slicing T and spatial coordinates L

$$
\begin{aligned}
\eta &= \tilde\eta + T\,, \\
x^i &= \tilde x^i + LY^i\,,
\end{aligned} \tag{A8}
$$

the metric variables transform as

$$
\begin{aligned}
A &= \tilde A - aH(T' + T)\,, \\
B &= \tilde B + aH(L' + k_H T)\,, \\
H_L &= \tilde H_L - aH(T + \frac{1}{3}k_H)\,, \\
H_T &= \tilde H_T + aH k_H L\,,
\end{aligned} \tag{A9}
$$